



Dynamics on Trees of Spheres

Matthieu Arfeux

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Dynamics on trees of spheres.

Matthieu Arfeux

June 24, 2014

Abstract

We introduce the notion of dynamically marked rational maps. We study sequences of analytic conjugacy classes of rational maps which diverge in moduli space. In particular, we are interested in the notion of rescaling limits introduced by Jan Kiwi. In order to deal with those, we introduce the notion of dynamical covers between trees of spheres for which a periodic sphere corresponds to a rescaling limit. We then recover results of Jan Kiwi regarding rescaling limits.

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Introduction.

The study of rescaling limits is a pretext for the introduction of the notion of cover between trees of spheres and dynamical systems of trees of spheres. These tools already appears hidden at the intersection of many different works. For example:

- in [Ch], [HK], [Ko] and [S] in the context of application of Thurston's results about the characterization of post critically finite topological dynamical covers that are realizable as rational maps and the study of the Teichmüller space;
- in [DMc], [S1] and [S2] where the authors associate trees to encode dynamical systems;
- in the use of Berkovich spaces in the context of holomorphic dynamics such as [K3], [DF] and [BKM].

This list is of course not exhaustive. The time offered by PhD seemed to be a good opportunity to fixe a flexible vocabulary that may unify these works. This paper is the first on a list of three papers related to [A]. This paper gives all the principal tools. The second one [A1] completes to the study of rescaling limits. The third one [A2] introduces the spaces of isomorphism classes of these tools and their natural topology.

Let us denote by $\mathbb{S} := P^1(C)$ the Riemann sphere. According to the uniformisation Theorem, every compact surface of genus 0 with a projective structure is isomorphic to \mathbb{S} . For $d \geq 1$, we denote by Rat_d the set of rational maps $f : \mathbb{S} \rightarrow \mathbb{S}$ of degree d . In particular, $\text{Aut}(\mathbb{S}) := \text{Rat}_1$ is the set of Moebius transformations. This set acts on Rat_d by conjugacy :

$$\text{Aut}(\mathbb{S}) \times \text{Rat}_d \ni (\phi, f) \mapsto \phi \circ f \circ \phi^{-1} \in \text{Rat}_d.$$

The quotient rat_d of Rat_d by this action is not compact and some interesting phenomena arise at its boundary.

Consider a diverging family of conjugacy classes of rational maps in rat_d . For example the family $f_\varepsilon : z \mapsto \varepsilon(z + 1/z)$ diverges in rat_d as ε tends to zero. For these representatives, we have $f_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ but after taking the second iterate we can see that it has a non-constant limit:

$$f_\varepsilon^2(z) = \varepsilon \left(\varepsilon(z + 1/z) + \frac{1}{\varepsilon(z + 1/z)} \right) \xrightarrow{\varepsilon \rightarrow 0} \frac{z}{z^2 + 1}.$$

More generally, consider a diverging sequence of conjugacy classes of rational maps in rat_d . The limits of sub-sequences of representatives $(f_n)_n$ are constant maps or maps with degree strictly less than d . Sometimes we can have an

integer $k \geq 1$ such that $(f_n^k)_n$ converges to a function f which is not constant (so dynamically interesting) even if every sub-sequence converges to a constant.

Jan Kiwi wrote a nice paper [K3] where he gives a lots of example of such behaviors and a nice historic on this topic. For his study he uses the formalism of Berkovich spaces in the continuity of [R], [K1] and [K2]. Following Jan Kiwi, we define **rescaling limits** as follows.

Definition. *For a sequence of rational maps $(f_n)_n$ of a given degree, a rescaling is a sequence of Moebius transformations $(M_n)_n$ such that there exist $k \in \mathbb{N}$ and a rational map g of degree ≥ 2 such that*

$$M_n \circ f_n^k \circ M_n^{-1} \rightarrow g$$

uniformly in compact subsets of \mathbb{S} with finitely many points removed.

If this k is minimum then it is called the rescaling period for $(f_n)_n$ at $(M_n)_n$ and g a rescaling limit for $(f_n)_n$.

Note that naturally we are interested in sequences in the quotient rat_d . That's why we define an equivalence relation associated to rescalings in order to look at rescaling limits in the natural quotient space $([g] \in \text{rat}_{\deg g})$ which is the one defined below.

Definition (Independence and equivalence of rescalings). *Two rescalings $(M_n)_n$ and $(N_n)_n$ of a sequence of rational maps $(f_n)_n$ are independent if $N_n \circ M_n^{-1} \rightarrow \infty$ in Rat_1 . That is, for every compact set K in Rat_1 , the sequence $N_n \circ M_n^{-1} \notin K$ for n big enough. They are said equivalent if $N_n \circ M_n^{-1} \rightarrow M$ in Rat_1 .*

We will write in this case $N_n \sim M_n$.

Again, following Jan Kiwi, we define the notion of dynamical dependence of rescalings.

Definition (Dynamical dependence). *Given a sequence $(f_n)_n \in \text{Rat}_d$ and given $(M_n)_n$ and $(N_n)_n$ of period dividing q . We say that $(M_n)_n$ and $(N_n)_n$ are dynamically dependent if, for some subsequences $(M_{n_k})_{n_k}$ and $(N_{n_k})_{n_k}$, there exist $1 \leq m \leq q$, finite subsets S_1, S_2 of \mathbb{S} and non constant rational maps g_1, g_2 such that*

$$L_{n_k}^{-1} \circ f_{n_k}^m \circ M_{n_k} \rightarrow g_1$$

uniformly on compact subsets of $\mathbb{S} \setminus S_1$ and

$$M_{n_k}^{-1} \circ f_{n_k}^{q-m} \circ L_{n_k} \rightarrow g_2$$

uniformly on compact subsets of $\mathbb{S} \setminus S_2$.

In this context Jan Kiwi proved the following result.

Theorem A. [K3] *For every sequence in Rat_d for $d \geq 2$ there are at most $2d - 2$ classes of dynamically independent rescalings with a non post-critically finite rescaling limit.*

We will reprove this result, using a different approach based on trees of spheres.

Outline.

In section 1, we define the notion of cover between trees of spheres. Let X be a finite set with at least 3 elements. A (stable) tree of spheres \mathcal{T} marked by X is the following data :

- a combinatorial tree T whose leaves are the elements of X , and whose internal vertices have at least valence 3 (stability), and
- for each internal vertex v of T , an injection $i_v : E_v \rightarrow \mathcal{S}_v$ of the set of edges E_v adjacent to v into a topological sphere \mathcal{S}_v .

We use the notation $X_v := i_v(E_v)$ and define the map $a_v : X \rightarrow \mathcal{S}_v$ such that $a_v(x) := i_v(e)$ if x and e lie in the same connected component of $T - \{v\}$.

This is a generalization of the notion of spheres marked by X defined below.

Definition (Marked sphere). *A sphere marked (by X) is an injection*

$$x : X \rightarrow \mathbb{S}.$$

We identify trees with only one internal vertex with the marked spheres. In the same spirit we generalize the notion of rational maps marked by a portrait defined below :

Definition (Marked rational maps). *A rational map marked by the portrait \mathbf{F} is a triple (f, y, z) where*

- $f \in \text{Rat}_d$
- $y : Y \rightarrow \mathbb{S}$ and $z : Z \rightarrow \mathbb{S}$ are marked spheres,
- $f \circ y = z \circ F$ on Y and
- $\deg_{y(a)} f = \deg(a)$ for $a \in Y$.

Where a portrait \mathbf{F} of degree $d \geq 2$ is a pair (F, \deg) such that

- $F : Y \rightarrow Z$ is a map between two finite sets Y and Z and
- $\deg : Y \rightarrow \mathbb{N} - \{0\}$ is a function that satisfies

$$\sum_{a \in Y} (\deg(a) - 1) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \deg(a) = d \quad \text{for all } b \in Z.$$

If (f, y, z) is marked by \mathbf{F} , we have the following commutative diagram :

$$\begin{array}{ccc} Y & \xrightarrow{y} & \mathbb{S} \\ F \downarrow & & \downarrow f \\ Z & \xrightarrow{z} & \mathbb{S}. \end{array}$$

Typically, $Z \subset \mathbb{S}$ is a finite set, $F : Y \rightarrow Z$ is the restriction of a rational map $F : \mathbb{S} \rightarrow \mathbb{S}$ to $Y := F^{-1}(Z)$ and $\deg(a)$ is the local degree of F at a . In this case, the Riemann-Hurwitz formula and the conditions on the function \deg implies that Z contains the set V_F of the critical values of F so that $F : \mathbb{S} - Y \rightarrow \mathbb{S} - Z$ is a cover.

Our generalization of marked rational maps is the notion of cover between trees of spheres. A cover $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ between two trees of spheres marked respectively by Y and Z is the following data

- a map $F : T^Y \rightarrow T^Z$ mapping leaves to leaves, internal vertices to internal vertices, and edges to edges,
- for each internal vertex v of T^Y and $w := F(v)$ of T^Z , a ramified cover $f_v : \mathcal{S}_v \rightarrow \mathcal{S}_w$ that satisfies the following properties:
 - the restriction $f_v : \mathcal{S}_v - Y_v \rightarrow \mathcal{S}_w - Z_w$ is a cover,
 - $f_v \circ i_v = i_w \circ F$,
 - if e is an edge between v and v' , then the local degree of f_v at $i_v(e)$ is the same as the local degree of $f_{v'}$ at $i_{v'}(e)$.

We will see that a cover between trees of spheres \mathcal{F} has a global degree, denoted by $\deg(\mathcal{F})$.

In section 2, we suppose in addition that $X \subseteq Y \cap Z$ and we show that we can associate a dynamical system to some covers between trees of spheres. More precisely we will say that \mathcal{F} is a dynamical system of trees of spheres and write $(\mathcal{F})_X$ if :

- $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between (stable) trees of spheres,
- there exists \mathcal{T}^X is a (stable) tree of spheres compatible with \mathcal{T}^Y and \mathcal{T}^Z , ie :
 - $X \subseteq Y \cap Z$
 - each internal vertex v of T^X is an internal vertex common to T^Y and T^Z ,
 - $\mathcal{S}_v^X = \mathcal{S}_v^Y = \mathcal{S}_v^Z$ and
 - $a_v^X = a_v^Y|_X = a_v^Z|_X$.

We will see that if such a \mathcal{T}^X exists then it is unique. With this definition we are able to compose covers along an orbit of vertices as soon as they are in T^X . When it is well defined we will denote by f_v^k the composition

$$f_{F^{k-1}(v)} \circ \dots \circ f_{F(v)} \circ f_v.$$

Dynamical covers between marked spheres can be naturally identified to dynamically marked rational maps:

Definition (Dynamically marked rational map). *A rational map dynamically marked by (\mathbf{F}, X) is a rational map (f, y, z) marked by \mathbf{F} such that $y|_X = z|_X$.*

We denote by $\text{Rat}_{\mathbf{F}, X}$ the set of rational maps dynamically marked by (\mathbf{F}, X) .

Let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system between trees of spheres. A period $p \geq 1$ cycle of spheres is a collection of spheres $(\mathcal{S}_{v_k})_{k \in \mathbb{Z}/p\mathbb{Z}}$ where the v_k are internal vertices of \mathcal{T}^X that satisfies $F(v_k) = v_{k+1}$. It is critical if it contains a critical sphere, ie a sphere \mathcal{S}_v such that $\deg(f_v)$ is greater or equal to two. If a sphere \mathcal{S}_v on a critical cycle contains a critical point of f_v that has infinite orbit, then the cycle is said non post-critically finite.

Using combinatorial and topological arguments, we prove the same type of results as Theorem A that can be expressed with this formalism in a weaker form as follows :

Theorem 1. *If $(\mathcal{F}, \mathcal{T}^X)$ is a dynamical system of topological trees of spheres then there are at most $2\deg(\mathcal{F}) - 2$ critical cycles of spheres which are not post-critically finite.*

In section 3, we consider holomorphic covers between trees of spheres with a projective structure: each sphere associated to an internal vertex has a projective structure and the covers between them are supposed to be holomorphic. We introduce the convergence notions of:

- a sequence of marked spheres to a marked tree of spheres. More precisely, a sequence x_n of marked spheres $x_n : X \rightarrow \mathbb{S}_n$ converges to a tree of spheres \mathcal{T}^X if for all internal vertex v of \mathcal{T}^X , there exists a projective isomorphism $\phi_{n,v} : \mathbb{S}_n \rightarrow \mathbb{S}_v$ such that $\phi_{n,v} \circ x_n$ converges to a_v . We will write $x_n \xrightarrow[\phi_n]{} \mathcal{T}^X$.
- a sequence of marked spheres covers $(f_n, y_n, z_n)_n$ to marked cover between trees of spheres $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$. We will write $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F}$. This notation means in particular that $y_n \xrightarrow[\phi_n^Y]{} \mathcal{T}^Y$ and $z_n \xrightarrow[\phi_n^Z]{} \mathcal{T}^Z$.
- a sequence of dynamical systems of marked spheres to a dynamical system of marked trees of spheres. We write $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}$ when the convergence is dynamical and we will have in particular $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F}$.

This convergence notion which is not Hausdorff comes from a topology. It will not be written in this article and can be found in [A] or will appear in [A2].

In section 4 we go back to the rescaling limits problem. We explain in which sense the existence of a periodic sphere for a dynamical cover between trees of spheres corresponds to the existence of a rescaling limit :

Theorem 2. *Let \mathbf{F} be a portrait, let $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F}, X}$ and let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres. Suppose that*

$$f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}.$$

If v is a periodic internal vertex in a critical cycle with exact period k , then $f_v^k : \mathbb{S}_v \rightarrow \mathbb{S}_v$ is a rescaling limit corresponding to the rescaling $(\phi_{n,v}^Y)_n$.

In addition, for every v' in the cycle, $(\phi_{n,v'}^Y)_n$ and $(\phi_{n,v}^Y)_n$ are dynamically dependent rescalings.

We then ask the reciprocal question. For this we recall the following famous result that follows from the Deligne-Mumford's compactification of the moduli space of stable curves :

Theorem B. [DM] *Given a finite set X with at least three elements, every sequence of spheres marked by X converges, after extracting a subsequence, to a tree of spheres marked by X .*

This theorem stated and proven in this terms in [A] and we will admit the following result that is already proven in [A] and will appear in [A2].

Theorem C. [A] *Let y_n, z_n be two sequences of spheres marked respectively by the finite sets Y and Z containing each one at least three elements and converging to the trees of spheres \mathcal{T}^Y and \mathcal{T}^Z .*

Every sequence of marked spheres covers $(f_n, y_n, z_n)_n$ of a given portrait converges to a cover between the trees of spheres \mathcal{T}^Y and \mathcal{T}^Z .

Then we are able to deduce that every rescaling limit can always appear on a dynamical cover between trees of spheres after choosing a good way to mark the respective sequence of rational maps:

Theorem 3. *Given a sequence $(f_n)_n$ in Rat_d for $(d \geq 2)$ with $p \in \mathbb{N}^*$ classes M_1, \dots, M_p of rescalings. Then, passing to a subsequence, there exists a portrait \mathbf{F} , a sequence $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F}, X}$ and a dynamical system between trees of spheres $(\mathcal{F}, \mathcal{T}^X)$ such that*

- $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}$ and
- $\forall i \in [1, p], \exists v_i \in \mathcal{T}^Y, M_i \sim (\phi_{n,v_i}^Y)_n$.

According to Theorems 2 and 3 we deduce Theorem A as a translation of Theorem 1.

Remark. *Note that all the objects introduced in this paper lie in three different categories:*

- *the category of combinatorial objects,*
- *the category of topological objects,*
- *the category of analytic objects.*

We also have sub categories for all of these ones where a special subset is marked and we can do dynamics. It might be interesting to keep this classification in mind all along the reading.

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1 Non dynamical objects

1.1 Combinatorial trees

1.1.1 Trees and sub-trees

Recall that a graph is the disjoint union of a finite set V called set of vertices and an other finite set E consisting of elements of the form $\{v, v'\}$ with $v, v' \in V$ called the set of edges. We say that $\{v, v'\}$ is an edge between v and v' . For all $v \in V$ we define E_v the set of vertices containing v . We call valence of v and denote by $\text{val}(v)$ the cardinal of E_v .

In a graph T , a path is a one-to-one map $t : [1, k] \rightarrow T$ such that for $j \in [1, k - 1]$,

- if $t(j)$ is a vertex, then $t(j + 1)$ is an edge and $t(j + 1) \in E_{t(j)}$ and
- if $t(j)$ is an edge, then $t(j + 1)$ is a vertex and $t(j) \in E_{t(j+1)}$.

We say that this path connects $t(1)$ to $t(k)$. We will do the confusion between a path and its image. We say that a path is connected if each vertex is connected to any other distinct one.

For a graph T , a cycle is a one-to-one map $t : \mathbb{Z}/k\mathbb{Z} \rightarrow T$ such that for $j \in [1, k - 1]$,

- if $t(j)$ is a vertex, then $t(j + 1)$ is an edge and $t(j + 1) \in E_{t(j)}$ and
- if $t(j)$ is an edge, then $t(j + 1)$ is a vertex and $t(j) \in E_{t(j+1)}$.

Definition 1.1 (Tree). *A tree is a connected graph without cycle.*

If a graph has no cycle than it is well known that there is always a unique path connecting two distinct vertices (see for example [Di, Theorem 1.5.1]). for a tree T we will denote by $[v_1, v_2]$ the unique path of T connecting v_1 to v_2 .

The path t will be denoted sometime by $[t(1), t(3), t(5), \dots, t(k)]$ if $t(1)$ and $t(k)$ are vertices or $]t(2), t(4), t(6), \dots, t(k - 1)[$ if $t(1)$ and $t(k)$ are edges.

A connected sub-graph of a tree T is a connected graph without cycle. It is a tree and we say that it is a sub-tree of T .

In a tree, vertices with valence 1 are called leaves. The other ones are called internal vertices. We denote by IV the set of Internal Vertices.

1.1.2 Topology

A graph T has a natural topology such that the closed sets are unions of sub-trees.

Definition 1.2 (Connected component). *The connected component of a sub-graph $T' \subset T$ is the connected sub-graph of T that is maximal for the inclusion.*

Definition 1.3 (Branch). *For v a vertex of a tree T and for $\star \in T - \{v\}$, a branch of \star on v is the connected component of $T - \{v\}$ containing \star . It is denoted by $B_v(\star)$.*

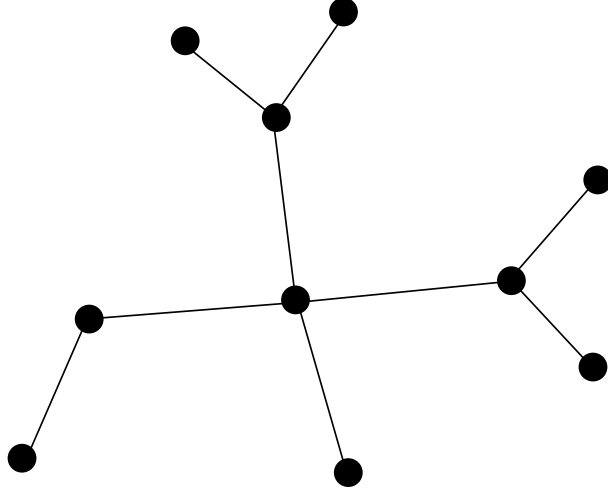


Figure 1: On this example, vertices are represented as dots and we represent by a segment an edge connecting the corresponding vertices.

Let $v \in V$. As T is a tree, for all element $\star \in T - \{v\}$, there is a unique path connecting v to \star . By definition this path contains a unique edge $e \in E_v$ so each branch on v will be denoted $B_v(e)$ with $e \in E_v$.

1.1.3 Characteristic

In the following, we introduce a tool called characteristic which is similar to the Euler characteristic and will be useful when we will talk about covers between trees of spheres. We will have a Riemann-Hurwitz formula.

Definition 1.4 (Characteristic of a sub-graph). *The characteristic of a vertex v of a graph T is*

$$\chi_T(v) := 2 - \text{val}(v).$$

The characteristic of a sub-graph T' of T is the integer

$$\chi_T(T') := \sum_{v \in V \cap T'} \chi_T(v).$$

We will simply use the notation $\chi(T')$ when it will not be confusing. Cf Figure 3 for an example.

Lemma 1.5. *For any tree T , we have $\chi_T(T) = 2$.*

Proof. Observe first that on a graph, each vertex v is connected to $\text{val}(v)$ edges and that each edge is connected to two vertices. Then we have

$$\sum_{v \in V} \text{val}(v) = 2\text{card}(E).$$

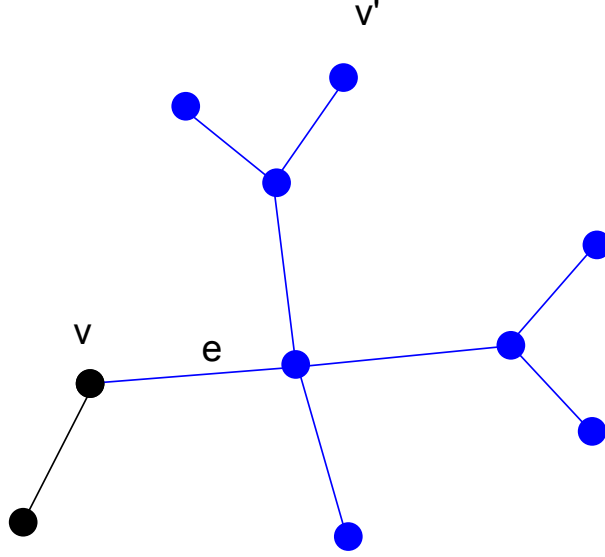


Figure 2: On this example the branch $B_v(e)$ (or $B_v(v')$) is colored in blue.

Moreover, in a tree, we have $\text{card}V = \text{card}E + 1$ (see [Di, corollary 1.5.3] for example). It follows that

$$\chi_T(T) = \sum_{v \in V} (2 - \text{val}(v)) = 2\text{card}V - 2\text{card}E = 2.$$

□

Recall that the adherence of a set is the smallest closed set containing it (cf figure 3).

Definition 1.6. If $T' \subseteq T$, we denote by

- $\overline{T'}$ the adherence of T' in T and
- $\partial_T T' := \overline{T'} - T'$ the frontiere of T' in T .

Lemma 1.7. If T' is open and connected in T , then the boundary $\partial_T T'$ is the set of vertices $v \in T - T'$ lying to an edge of T' . The adherence $\overline{T'}$ is a sub-tree of T for which the set of internal vertices is $IV \cap T'$.

Proof. The adherence of T' is the smallest sub-graph of T containing T' . It has to contain all vertices $v \in T$ lying to an edge of T' . It is not necessary to add other vertices or other edges in order to obtain a graph. This proves that $\partial_T T'$ is the set of vertices $v \in T - T'$ lying to an edge of T' .

The adherence of a connected open set is a sub-graph of T . So it is a sub-tree of T . The vertices of $\partial_T T'$ are the leaves of $\overline{T'}$. If it is not the case then

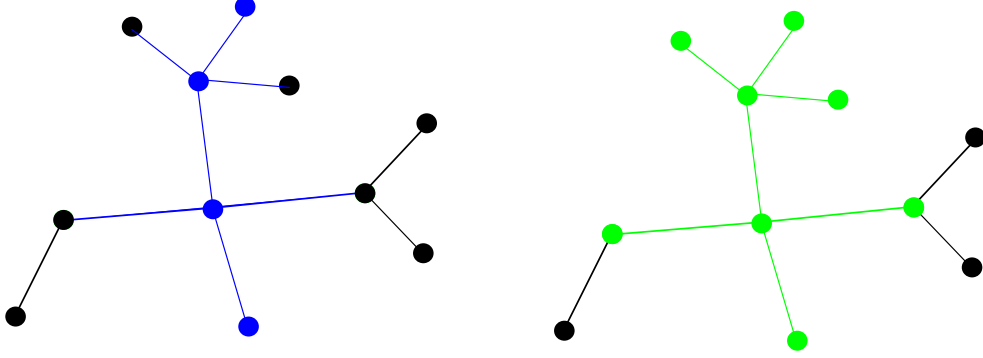


Figure 3: An open connected sub-graph of T' in blue of characteristic -2 to the left and \overline{T}' in green to the right.

$T' = \overline{T}' - \partial_T T'$ would not be connected. The set T' is open, so for all vertex v of T' , we have $E_v \subset T'$. Consequently the valence of v in \overline{T}' is the same as the one of v in T . This proves that internal vertices of \overline{T}' are internal vertices of T contained in T' . \square

Lemma 1.8. *If T' is a non empty sub-graph of T , open and connected, then*

$$\chi_T(T') = 2 - \text{card} \partial_T T'.$$

Proof. In \overline{T}' , each vertex v of T' has valence $\text{val}(v)$ and each vertex of $\partial_T T'$ has characteristic 1. According to lemma 1.5, we conclude that

$$2 = \chi_T(\overline{T}') = \sum_{v \in V \cap T'} \chi_T(v) + \sum_{v \in \partial_T T'} \chi_T(v) = \chi_T(T') + \text{card} \partial_T T'.$$

\square

Lemma 1.9. *If T' is a non empty sub-graph of T , open and connected, then*

1. $\chi_T(T') \leq 2$;
2. $\chi_T(T') = 2$ iff $T' = T$;
3. $\chi_T(T') = 1$ iff T' is a branch of T .

Proof. According to the previous lemma, $\chi_T(T') = 2 - \text{card} \partial_T T'$.

1) evident.

2) $\chi_T(T') = 2$ iff $\partial_T T' = \emptyset$ iff T' is open and closed iff $T' = T$.

3) If T' is a branch we have vertex v , then $\partial_T T' = \{v\}$ and $\chi_T(T') = 1$. Reciprocally, if $\chi_T(T') = 1$, then $\partial_T T'$ contains a unique vertex v . Let $e := \{v, v'\}$ be the edge of T' containing v and define $B := B_v(e)$. As T' is connected, contained in $T - \{v\}$ and contains e , we have $T' \subseteq B$. Given that $T' \cap B = \emptyset$,

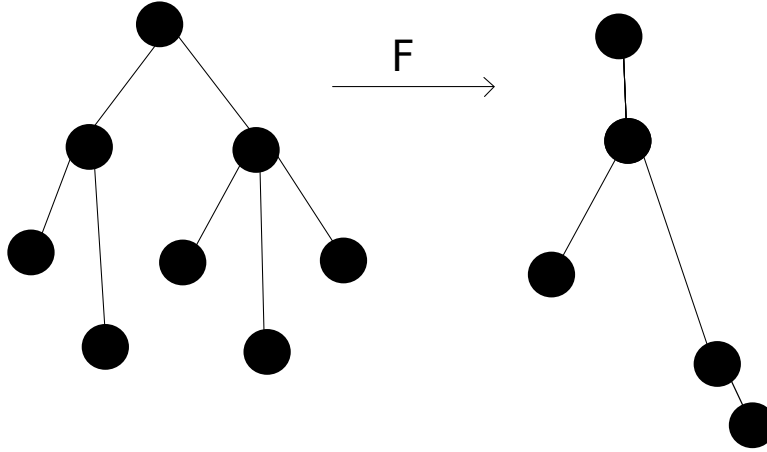


Figure 4: Example of a trees cover: the image of a vertex is the vertex at the same horizontal level. In this example the map is not surjective.

the branch B is the disjoint reunion of two open sets T' and $B - \overline{T'} = B - T'$. As B is connected, we have $B - T' = \emptyset$ and it follows that $B = T'$. \square

1.2 Combinatorial trees maps

Definition 1.10 (Trees map). *A map $F : T \rightarrow T'$ is a trees map if*

- *T and T' are trees;*
- *vertices map to vertices : $F(V) \subseteq V'$;*
- *every edge connecting two vertices maps to an edge connecting the image of these vertices : if $\{v, w\} \in E$, then $F(\{v, w\}) = \{F(v), F(w)\} \in E'$.*

Let observe that if U is a sub-graph of T , then $F(U)$ is a sub-graph of T' and, inversely, if U' is a sub-graph of T' , then $F^{-1}(U')$ is a sub-graph of T . Particularly, the preimage of closed sets are closed:

Proposition 1.11. *Trees maps are continuous and the image of a sub-tree is a sub-tree.*

Proof. A connected set maps to a connected one. \square

1.3 Trees of spheres

Thereafter, X , Y and Z will design finite sets with at least 3 elements.

Definition 1.12 (Marked tree). *A tree T marked by X is a tree such that the leaves are the elements of X .*

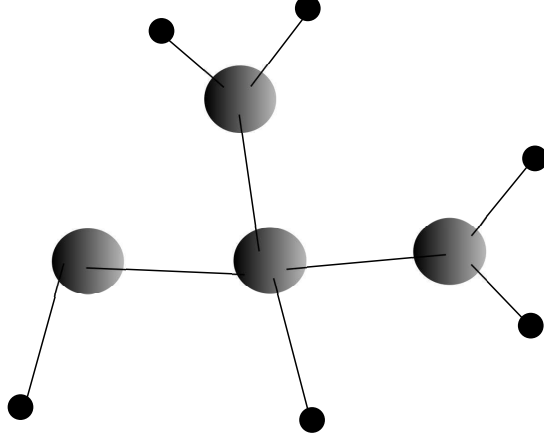


Figure 5: A tree of spheres having the same combinatorial tree as the one on figure 2.

A tree marked by X will be denoted by T^X . Every object Obj referring to T^* will be denoted by Obj^* . For example E^Y is the edges set of T^Y .

Definition 1.13 (Marked tree of spheres). *A (topological) tree of spheres \mathcal{T}^X (marked by X) is the data of:*

- a combinatorial tree T^X and
- for every internal vertex v of T^X ,
 - a topological sphere \mathcal{S}_v and
 - a one-to-one map $i_v : E_v \rightarrow \mathcal{S}_v$.

For $e \in E_v$, we say that $i_v(e)$ is the attaching point of e on v . We will often use the notation $e_v := i_v(e)$ sometime even $i_v(v') := e_v$ if $v' \in B_v(e)$. We define $X_v := i_v(E_v)$ the set of attaching points on the sphere \mathcal{S}_v .

Remark 1.14. Giving a one-to-one map $i_v : E_v \rightarrow \mathcal{S}_v$, is the same as giving a map $a_v : X \rightarrow \mathcal{S}_v$ such that $a_v(x_1) = a_v(x_2)$ if and only if x_1 and x_2 are in the same corresponding branch of v . This means $a_v(x) := i_v(e)$ if x lies in $B_v(e)$.

Example. [Marked spheres] A tree of spheres marked by X with a unique internal vertex v is the same data as this vertex v and the map i_v . We call it a marked sphere or a sphere marked by X .

1.4 Covers between trees of spheres

1.4.1 Definitions and degree

A cover between trees of spheres is the extension of the notion of combinatorial trees cover to trees of spheres. We add the data of a ramified cover for each

internal vertex and require that the ramification locus is contained in the set of the edges attaching points.

Definition 1.15 (Cover). *A cover between trees of spheres $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is the following data :*

- a trees map $F : T^Y \rightarrow T^Z$ mapping leaves to leaves and internal vertices to internal vertices ($F(Y) \subseteq Z$ and $F(IV^Y) \subseteq IV^Z$) and
- for every internal vertex $v \in IV^Y$ and $w := F(v) \in IV^Z$, a topological ramified cover $f_v : \mathcal{S}_v \rightarrow \mathcal{S}_w$ such that
 1. the restriction $f_v : \mathcal{S}_v - Y_v \rightarrow \mathcal{S}_w - Z_w$ is a cover;
 2. $f_v \circ i_v = i_w \circ F$ on E_v ;
 3. if $e = \{v_1, v_2\} \in E^Y$ is an edge connecting two internal vertices, then $\deg_{e_{v_1}} f_{v_1} = \deg_{e_{v_2}} f_{v_2}$.

Example. [Spheres covers] A cover between trees of spheres $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ such that \mathcal{T}^Y and \mathcal{T}^Z are marked spheres (with respective unique internal vertices v and v') is the same data as a ramified cover between \mathbb{S}_v and $\mathbb{S}_{v'}$ such that the set of attaching points on \mathbb{S}_v is the pre-image of the set of attaching points on $\mathbb{S}_{v'}$ and contains the ramification locus. We says that it is a marked spheres cover and do the confusion between \mathcal{F} and $(f_v, a_v^Y, a_{v'}^Z)$.

For every internal vertex $v \in IV^Y$, we define $\deg(v) := \deg(f_v)$ to simplify the expression. As well, for all $x \in \mathcal{S}_v$ we define $\deg(x) := \deg_x f_v$. The condition 3 assures that we can define a degree for every edge e connecting two internal vertices v_1 and v_2 of T^Y , that will be denoted by

$$\deg(e) := \deg_{e_{v_1}} f_{v_1} = \deg_{e_{v_2}} f_{v_2}.$$

Each leaf $y \in Y$ is connected to a unique internat vertex v by an edge e , so we can define

$$\deg(y) := \deg(e) := \deg_{e_v} f_v.$$

This define a degree map for the map $F : Y \rightarrow Z$.

Definition 1.16. *A critical vertex (resp. critical leaf) of \mathcal{F} is a vertex of T^Y (resp. a leaf $y \in Y$) having degree more than one. We then define $\text{mult}(y) := \deg(y) - 1$, the multiplicity of y . We denote by $\text{Crit}\mathcal{F}$ the set of critical leaves of \mathcal{F} .*

For each vertex v of T^Z and each leave e of T^Z we can define

$$D_v := \sum_{v' \in F^{-1}(v)} \deg(v') \quad \text{and} \quad D_e := \sum_{e' \in F^{-1}(e)} \deg(e').$$

Lemma 1.17. *If $e \in E_v$, then $D_e = D_v$.*

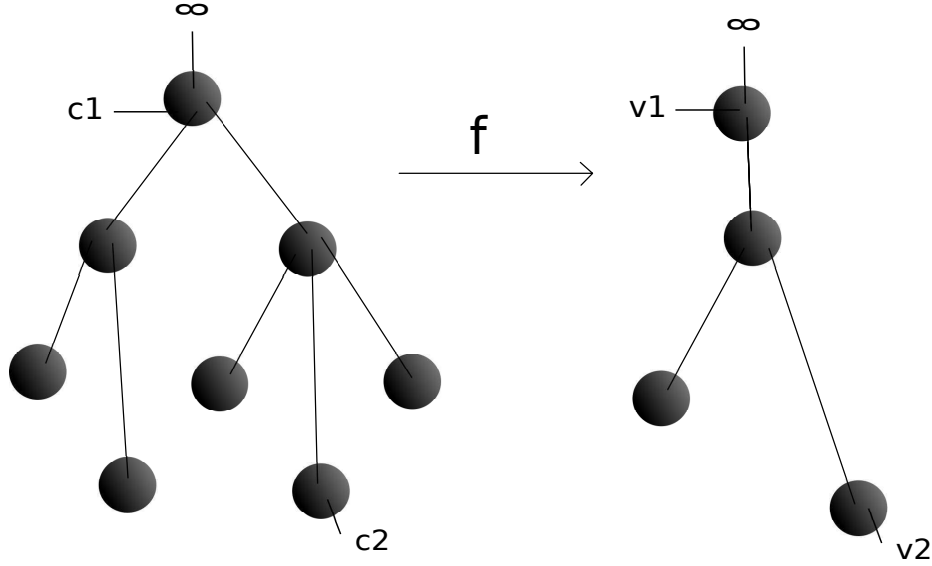


Figure 6: A cover between trees of spheres of degree 3. The sphere at the top on the left maps by a cover of the type z^3 , the two spheres connecting $c2$ to this one are maps by a cover of same type to their images. The others are maps by an identity type cover.

Proof. If v is a leaf, then preimages v' of v are leaves on which are attached preimages e' of e . The lemma is clear because by definition $\deg(v') = \deg(e')$.

If v is an internal vertex, set X the set of points x lying in the sphere $\mathcal{S}_{v'}$ with $F(v') = v$ and $f_{v'}(x) = e_v$. Let $x \in X$. Given that $e_v \in Z_v$ and that $f_{v'} : \mathcal{S}_{v'} - Y_{v'} \rightarrow \mathcal{S}_v - Z_v$ is a cover, then $x \in Y_{v'}$. Consequently, x is the attaching point of an edge e' of T^Y mapped to e . Inversely, if $F(e') = e$, then e' is attached to a sphere $v' \in F^{-1}(v)$ at a point $x \in X$. So we have

$$\begin{aligned} D_e &= \sum_{e' \in F^{-1}(e)} \deg(e') = \sum_{x \in X} \deg(x) \\ &= \sum_{v' \in F^{-1}(v)} \sum_{x \in f_{v'}^{-1}(e_v)} \deg(x) = \sum_{v' \in F^{-1}(v)} \deg(v') = D_v. \end{aligned}$$

□

Thus if e is an edge connecting two vertices v and w , then $D_v = D_e = D_w$. This number is constant, because the tree T^Z is connected. It does not depend on e neither on v . We denote by D this number and call it the degree of \mathcal{F} .

Corollary 1.18. *The map $F : T^Y \rightarrow T^Z$ is surjective.*

Proof. For every vertex v of T^Z , we have $D_v \neq 0$.

□

The following lemma and its corollary help to visualize the set of critical vertices distribution on a tree.

Lemma 1.19. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres. Every critical vertex lies in a path connecting two critical leaves. Each vertex on this path is critical.*

Proof. Let v be a critical vertex of \mathcal{F} . Then f_v has at least two distinct critical points. There are at least two distinct edges attached to v . So v is on a path of critical vertices.

Let $[v_1, v_2]$ be such a path with a maximal number of vertices. From this maximality property, we see that there is only one critical edge (edge with degree strictly greater than one) attached to v_1 . If v_1 is not a leaf then f_{v_1} has just one critical point and that is not possible. So v_1 is a leaf. As well, v_2 is a leaf. \square

Recall that the characteristic of a vertex v of T^X is $\chi_{T^X}(v) := 2 - \text{val}(v)$, and thus it is equal to the Euler characteristic of $\mathcal{S}_v - X_v$.

We have a natural Riemann-Hurwitz formula for covers between trees of spheres where $\chi_{\mathcal{T}}$ plays the same role as the Euler characteristic.

Proposition 1.20 (Riemann Hurwitz formula). *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres of degree D . Let T'' be a sub-graph of T^Z and $T' := F^{-1}(T'')$. Then*

$$\chi_{T^Y}(T') = D \cdot \chi_{T^Z}(T'') - \sum_{y \in \text{Crit} \mathcal{F} \cap T'} \text{mult}(y).$$

Proof. If $v'' \in IV^Z$, then from the Riemann-Hurwitz formula we have

$$\sum_{v' \in F^{-1}(v'')} \chi_{T^Y}(v') = \sum_{v' \in F^{-1}(v'')} \deg(v') \cdot \chi_{T^Z}(v) = D \cdot \chi_{T^Z}(v''). \quad (1)$$

Otherwise, a leaf has characteristic 1, so for every leaf y of T^Y , we have

$$\chi_{T^Y}(y) = \deg(y) - \text{mult}(y).$$

Then, for every leaf $z \in Z$, we deduce that

$$\sum_{y \in F^{-1}(z)} \chi_{T^Y}(y) = \sum_{y \in F^{-1}(z)} \deg(y) - \text{mult}(y) = D \cdot \chi_{T^Z}(z) - \sum_{y \in F^{-1}(z)} \text{mult}(y). \quad (2)$$

By adding (1) and (2) for all vertices $v'' \in IV^Z \cap T''$ and leaves $z \in T'' \cap Z$, we get the formula. \square

1.4.2 Consequences of the Riemann-Hurwitz formula

Corollary 1.21. *If $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between trees of spheres of degree D , then the tree T^Y has $2D - 2$ critical leaves counted with multiplicities.*

Proof. We use the Riemann-Hurwitz Formula for $T' = T^Y$ and $T'' = T^Z$, and the fact that $\chi_{T^Y}(T^Y) = \chi_{T^Z}(T^Z) = 2$ (cf lemma 1.5.) \square

Recall. A portrait \mathbf{F} of degree $d \geq 2$ is a pair (F, \deg) where

- $F : Y \rightarrow Z$ is a map between finite sets Y and Z and
- $\deg : Y \rightarrow \mathbb{N} - \{0\}$ is a map that verifies

$$\sum_{a \in Y} (\deg(a) - 1) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \deg(a) = d \quad \text{for all } b \in Z.$$

We proved that the pair $(F|_Y, \deg_{F|_Y})$ defines a portrait.

Corollary 1.22. *If $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between trees of spheres of degree D , then*

$$2 - \text{card}(Y) = D \cdot (2 - \text{card}(Z)).$$

Proof. We apply the Riemann-Hurwitz formula for $T' = T^Y - Y$ and $T'' = T^Z - Z$, using the fact that $\chi_{T^Y}(T') = 2 - \text{card}Y$ and $\chi_{T^Z}(T'') = 2 - \text{card}(Z)$ (cf lemma 1.8). Given that T' doesn't have any leaf of T^Y , this proves the result. \square

We proved that the degree of \mathcal{F} is bounded relatively to $\text{card}(Y)$ and $\text{card}(Z)$.

Lemma 1.23. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres. Let T'' be an open, non empty and connected subset of T^Z and let T' be a connected component of $F^{-1}(T'')$. Then the map $\overline{\mathcal{F}} : \overline{\mathcal{T}}' \rightarrow \overline{\mathcal{T}}''$ defined by*

- $\overline{F} := F : \overline{\mathcal{T}}' \rightarrow \overline{\mathcal{T}}''$ and
- $\overline{f}_v := f_v$ if $v \in V' - Y'$

is a cover between trees of spheres.

Proof. Indeed for every vertex $v \in V' - Y'$, edges on v in $\overline{\mathcal{T}}'$ are the same as the one on T so \overline{f}_v satisfies the required conditions. Moreover, leaves of $\overline{\mathcal{T}}'$ are either leaves of T and map to leaves of T^Z , so leaves of $\overline{\mathcal{T}}''$ or are elements of $\overline{\mathcal{T}}' - T'$. In this case they map to elements of $\overline{\mathcal{T}}'' - T''$ which are leaves of $\overline{\mathcal{T}}''$ because adjacent vertices are mapped to adjacent vertices. \square

We define $\deg(\mathcal{F}|_{T'_Y}) := \deg \overline{\mathcal{F}}$ and $\text{mult } T'_Y := \deg(\mathcal{F}|_{T'_Y}) - 1$.

Then we have the restriction of the Riemann-Hurwitz formula to a connected component of the preimage.

Proposition 1.24. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres. Let T'' be a sub-tree of T^Z . Let T' be a connected component of $F^{-1}(T'')$. Then we have*

$$\chi_{T^Y}(T') = \deg(\mathcal{F}|_{\overline{\mathcal{T}}'}) \cdot \chi_{T^Z}(T'') - \sum_{y \in \text{Crit } \mathcal{F} \cap T'} \text{mult}(y).$$

Proof. Given that $\chi_{T^Y}(T') = \chi_{\widehat{T}'}(T')$ and $\chi_{T^Z}(T'') = \chi_{\widehat{T}''}(T'')$, the result follows immediately by using the Riemann-Hurwitz formula on the cover $\overline{\mathcal{T}} : \overline{\mathcal{T}}' \rightarrow \overline{\mathcal{T}}''$. \square

2 Dynamics on stable trees

In this section we suppose that $X \subseteq Y \cap Z$.

2.1 Stable tree and dynamical system

Definition 2.1 (Stable tree). *A tree T is stable if every vertex has valence greater than three.*

From here until the end of the article we suppose that **trees are stable**.

Definition 2.2. *In a tree T , we say that a vertex v separates three vertices v_1 , v_2 and v_3 if the v_i are in distinct connected components of $T - \{v\}$.*

Note that three distinct vertices of T lie either on a same path or they are separated by a unique vertex.

Definition 2.3 (Compatible tree). *A tree T^X is compatible with a tree T^Y if*

- $X \subseteq Y$, $IV^X \subseteq IV^Y$ and
- for all vertices v , v_1 , v_2 and v_3 of V^X , the vertex v separates v_1 , v_2 and v_3 in T^X if and only if it does the same in T^Y .

Later in the article, it will be useful to know if a vertex is in T^X . The two following lemmas give a way to do this in some particular cases.

Lemma 2.4. *If T^X is compatible with T^Y and if an internal vertex $v \in IV^Y$ separates three vertices of V^X , then $v \in T^X$.*

Proof. Let v_1 , v_2 and v_3 be these three vertices. There is an internal vertex v^X of T^X separating v_1 , v_2 and v_3 in T^X . From the compatibility we conclude that this vertex separates v_1 , v_2 and v_3 in T^Y . It follows that $v^X = v$. \square

Now we focus on trees of spheres.

Definition 2.5. *A tree of spheres \mathcal{T}^X is compatible with a tree of spheres \mathcal{T}^Y if*

- \mathcal{T}^X is compatible with \mathcal{T}^Y ,
- for all internal vertex v of \mathcal{T}^X , we have
 - $\mathcal{S}_v^X = \mathcal{S}_v^Y$ and
 - $a_v^X = a_v^Y|_X$.

If it is the case we write $\mathcal{T}^X \triangleleft \mathcal{T}^Y$. Now we can define a dynamical system of trees of spheres. Note that when the spheres will be equipped with a projective structure, then we will require in addition that \mathcal{S}_v^X and \mathcal{S}_v^Y have the same one.

Definition 2.6 (Dynamical systems). *A dynamical system of trees of spheres is a pair $(\mathcal{F}, \mathcal{T}^X)$ such that*

- $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between trees of spheres ;
- $\mathcal{T}^X \triangleleft \mathcal{T}^Y$ and $\mathcal{T}^X \triangleleft \mathcal{T}^Z$.

In [A2] we prove that if such a \mathcal{T}^X exists then it is unique. Figure 7 gives such an example of dynamical system.

Example. [Spheres dynamical system] Let $(\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z, \mathcal{T}^X)$ be a dynamical system such that \mathcal{F} is a cover of marked spheres. Then \mathcal{T}^X has a unique internal vertex and given that \mathcal{T}^Y and \mathcal{T}^Z have only one internal vertex, then the one of \mathcal{T}^X is the same as v the one of \mathcal{T}^Y and of \mathcal{T}^Z . Then we identify (f_v, a_v^Y, a_v^Z) and $(\mathcal{F}, \mathcal{T}^X)$. We say that it is a dynamical system of spheres marked by $\mathbf{F} := (F, \deg)$.

2.2 Dynamics on combinatorial trees

As we have a common set V^X in the trees \mathcal{T}^Y and \mathcal{T}^Z , we can try now to iterate F as soon as images stay in V^X . Recursively we define for $k \geq 1$

$$IV(F) := IV^X \quad \text{and} \quad IV(F^{k+1}) := \{v \in IV(F^k) \mid F^k(v) \in IV^X\}.$$

Define

$$\text{Prep}(F) := \bigcap_{k \geq 1} IV(F^k).$$

If $v \in \text{Prep}(F)$, then $F^k(v)$ is well defined and lies in IV^X for all $k \geq 0$.

The set $\text{Prep}(F)$ is finite and invariant under the map F , each vertex v of $\text{Prep}(F)$ is (pre)periodic under F . It may happen that $\text{Prep}(F)$ is empty as we can see on the example on figure 7.

If $v \in IV^Y - \text{Prep}(F)$, then there exists a smallest integer $k \in \mathbb{N}$ such that $F^k(v) \notin V^X$. We say that v is forgotten by F^k or simply that v is forgotten if $k = 0$. On figure 7, internal vertices of \mathcal{T}^Y at the bottom on \mathcal{T}^Y are forgotten by F^3 .

Restricting the dynamic on vertices would be ignoring the tree structure. The following lemma show a strong restriction coming from the compatibility.

Lemma 2.7. *Let $B \subset \mathcal{T}^Z$ be a branch on $v \in V^X$. If B contains a vertex of V^X then its attaching point $i_v^Z(B)$ lies in X_v .*

Proof. Either this vertex is a leaf and the result is trivial, or it is not a leaf and B contains a leaf of \mathcal{T}^X then we are in the previous case. \square

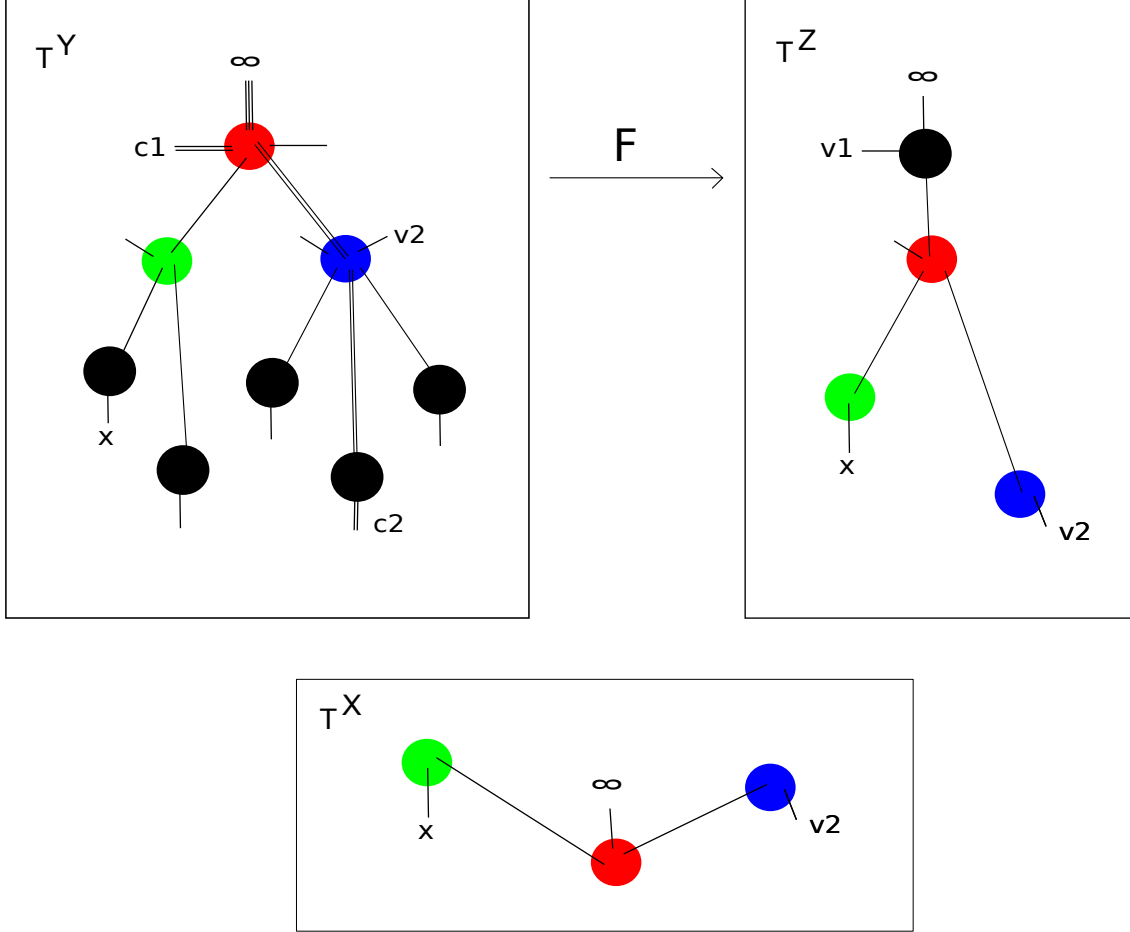


Figure 7: On this example, internal vertices which are not in \mathcal{T}^X are in black whereas internal vertices which are the same in T^X , T^Y and T^Z have the same color. The pair $(\mathcal{F}, \mathcal{T}^X)$ is a dynamical system. The internal vertex adjacent to c_2 maps to the blue vertex by a degree two cover. Then it maps to the red one by the same kind of cover. Then it maps to the higher black one on T^Z by a degree three cover. All other vertices map with degree one. On this example, each vertex cannot be iterated more than three times.

2.3 Dynamics on trees of spheres

If $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between trees of spheres, if $v \in T^Y$ then we will denote by f_v^k the composition $f_{F^{k-1}(v)} \circ \dots \circ f_{F(v)} \circ f_v$ as soon as $v \in IV(F^{k'})$ for some $k' > k$. We define Σ as the disjoint union of the \mathcal{S}_v for $v \in T^Y$. The orbit of any point z in Σ is the set

$$\mathcal{O}(z) := \{f_v^k(z) \mid k \geq 0, z \in \mathcal{S}_v, v \in IV(F^k)\}.$$

Two points of Σ are in the same grand orbit if their orbits intersect. We denote by $\mathcal{GO}(z)$ the set of points of Σ that are in the same grand orbit as z . We say that the orbit of a point z in Σ is infinite if $\text{card } \mathcal{O}(z)$ is not finite. We set \mathcal{GO}^∞ the set of infinite grand orbits containing a critical point.

Theorem 2.8 (Spheres periodic cycles). *Let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres. Then, $\text{card } \mathcal{GO}^\infty \leq 2\deg(\mathcal{F}) - 2$.*

Proof. Let $c \in \Sigma$ be a critical point of the map f such that $\mathcal{GO}(c) \in \mathcal{GO}^\infty$. Then c lies in a sphere \mathcal{S}_v with $v \in \text{Prep}(F)$. For $k \geq 0$, we define $v_k := F^k(v)$ and $c_k := f_v^k(c)$. We have $\text{card}\{c_k\} = \infty$, so there exists $k_0 \geq 1$ such that

- c_k is an attaching point of an edge in \mathcal{T}^Y for $k < k_0$ and
- c_{k_0} is not the attaching point of an edge of T^Y

(indeed, the number of edges attaching points in \mathcal{T}^Y is finite).

For $k \in [0, k_0 - 1]$, we define

- B_k^Y the branch of T^Y on v_k attached to c_k ,
- B_{k+1}^Z the branch of T^Z on v_{k+1} attached to c_{k+1} and
- $\tilde{B}_k = B_k^Y \cap F^{-1}(B_{k+1}^Z)$.

Let $k_1 \geq 1$ be the minimal integer such that $\tilde{B}_k = B_k^Y$ for $k \in [k_1, k_0 - 1]$. We define

$$B_c := \bigcup_{k_1-1}^{k_0-1} \tilde{B}_k.$$

Given that c_{k_0} is not an attaching point of T^Y , every vertex of B_c is forgotten by an iterate of F . In other words, $B_c \cap \text{Prep}(F) = \emptyset$.

Lemma. *The open set B_c contains a critical leaf.*

Proof. Either $k_1 = 0$ and $B_0^Y = \tilde{B}_0$. From the Riemann-Hurwitz formula, we have

$$\begin{aligned} 1 &= \chi_{T^Y}(\tilde{B}_0) = \deg(\mathcal{F} : \tilde{B}_0 \rightarrow B_1^Z) \cdot \chi_{T^Z}(B_1^Z) - \text{mult}(\tilde{B}_0) \\ &\geq \deg(\mathcal{F} : \tilde{B}_0 \rightarrow B_1^Z) - \text{mult}(B_c). \end{aligned}$$

Given that c is a critical point of f_v , we have

$$\deg(F : \tilde{B}_0 \rightarrow B_1^Z) \geq \deg(v) \geq 2.$$

So $\text{mult}(B_c) \geq 1$ and B_c contains at least a critical leaf.

We know that $k_1 \geq 1$ and that \tilde{B}_{k_1-1} is not a branch. According to lemma 1.9, we deduce that $\chi_{T^Y}(\tilde{B}_{k_1-1}) \leq 0$. From the Riemann-Hurwitz formula, we have

$$\begin{aligned} 0 &\geq \chi_{T^Y}(\tilde{B}_{k_1-1}) = \deg(\mathcal{F} : \tilde{B}_{k_1-1} \rightarrow B_{k_1}^Z) \cdot \chi_{T^Z}(B_{k_1}^Z) - \text{mult}(\tilde{B}_{k_1-1}) \\ &\geq 1 - \text{mult}(B_c). \end{aligned}$$

So $\text{mult}(B_c) \geq 1$ and B_c contains at least a critical leaf. \square

Lemma. *Let $c \in \Sigma$ and $c' \in \Sigma$ be two attaching points with infinite disjoint orbits. Then $B_c \cap B_{c'} = \emptyset$.*

Proof. If $B_c \cap B_{c'} \neq \emptyset$, then $F(B_c) \cap F(B_{c'}) \neq \emptyset$ and we can find two integers k and k' such that the branch of T^Z attached to c_k intersect the branch of T^Z attached to $c'_{k'}$. In this case,

- Either $v_k = v'_{k'}$ and $c_k = c'_{k'}$, which contradict the fact that orbits of c and c' are disjoint ;
- either v_k lies in the branch of T^Z attached to $v'_{k'}$. As $\text{Prep}(F) \cap V^Z \subset V^X$, we have $B_{c'} \cap \text{Prep}(F) = \emptyset$ which contradicts lemma 2.7 ;
- or $v'_{k'}$ lies in the branch of T^Z attached to v_k . As $\text{Prep}(F) \cap V^Z \subset V^X$, we have $B_c \cap \text{Prep}(F) = \emptyset$ which contradicts lemma 2.7 .

\square

Now we finish the theorem proof. Let c_1, \dots, c_N be critical points with disjoint and infinite orbits. The open sets B_{c_1}, \dots, B_{c_N} are disjoint and each one contains a critical leaf of T^Y . The number of critical leaves counted with multiplicity is $2\deg(\mathcal{F}) - 2$ so $N \leq 2\deg(\mathcal{F}) - 2$. \square

As a special case we deduce Theorem 1.

Theorem (1). *If $(\mathcal{F}, \mathcal{T}^X)$ is a dynamical system of topological trees of spheres then there are at most $2\deg(\mathcal{F}) - 2$ critical cycles of spheres which are not post-critically finite.*

3 Convergence notions

Recall that trees are supposed to be **stable**. Here we require that the trees are **projective** and that all the covers are **holomorphic** in a sense that we define below.

In this chapter, one define a notion of convergence on the set of trees of spheres. This notion is not Hausdorff but in [A2] I show that it corresponds to an Hausdorff topology on the natural quotient of this set under the action of trees of spheres isomorphisms.

3.1 Holomorphic covers

Definition 3.1 (Projective structure). *A projective structure on a tree of spheres \mathcal{T} marked by X is the data for every $v \in IV$ of a projective structure on \mathcal{S}_v .*

From the Uniformisation Theorem, it is the same as giving a complex structure on \mathcal{S}_v and giving a class of homeomorphisms $\sigma : \mathcal{S}_v \rightarrow \mathbb{S}$ where σ is equivalent to σ' when $\sigma' \circ \sigma^{-1}$ is a Moebius transformation. Such a σ is called a projective chart on \mathcal{S}_v . When the topological sphere \mathcal{S}_v has such a projective structure, we will denote it by \mathbb{S}_v .

Definition 3.2 (Holomorphic covers). *A cover between trees of spheres $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ with a given projective structure is holomorphic if for all internal vertex v , the map $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$ is holomorphic.*

If f_v is holomorphic then its expression in projective charts is a rational map.

When a tree of spheres is compatible to an other one we require that the projective structures on a common sphere is the same.

3.2 Convergence of marked spheres

Recall that a sphere marked by X is an injection $x : X \rightarrow \mathbb{S}$. Sometime we will do the confusion between x and the tree of sphere with only one internal vertex, the corresponding sphere being \mathbb{S} and the marking given by $x : X \rightarrow \mathbb{S}$.

Definition 3.3. *A sequence of marked spheres $x_n : X \rightarrow \mathbb{S}_n$ converges to a tree of spheres \mathcal{T}^X if for all internal vertex v of \mathcal{T}^X , there exists a (projective) isomorphism $\phi_{n,v} : \mathbb{S}_n \rightarrow \mathbb{S}_v$ such that $\phi_{n,v} \circ x_n$ converges to a_v .*

(We prefer to use the notation \mathbb{S}_n instead of \mathbb{S} because the \mathbb{S}_n can be distincts.) We will use the notation $x_n \rightarrow \mathcal{T}^X$ or $x_n \xrightarrow{\phi_n} \mathcal{T}^X$.

Example. Suppose that $X := \{\chi_1, \chi_2, \chi_3, \chi_4\}$. For all $n \geq 1$, let $x_n : X \rightarrow \mathbb{S}$ be the marked sphere defined by :

$$x_n(\chi_1) := 0, \quad x_n(\chi_2) := 1, \quad x_n(\chi_3) := n \quad \text{and} \quad x_n(\chi_4) := \infty.$$

Let \mathcal{T}^X be the tree of projective spheres marked by X with two distinct internal vertices v and v' of valence 3 with $\mathbb{S}_v := \mathbb{S}_{v'} := \mathbb{S}$,

$$a_v(\chi_1) := 0, \quad a_v(\chi_2) := 1, \quad a_v(\chi_3) := a_v(\chi_4) := \infty,$$

$$a_{v'}(\chi_1) := a_{v'}(\chi_2) := 0, \quad a_{v'}(\chi_3) := 1 \quad \text{and} \quad a_{v'}(\chi_4) := \infty.$$

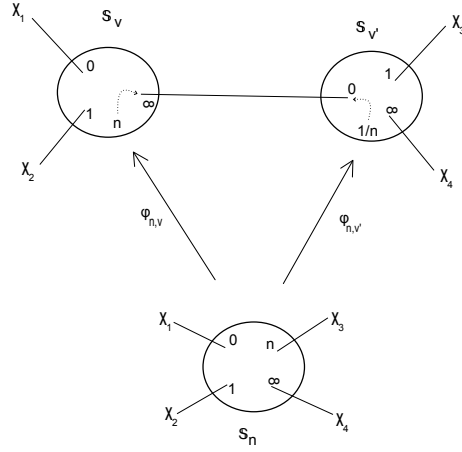


Figure 8:

Considering the isomorphisms $\phi_{n,v} : \mathbb{S} \rightarrow \mathbb{S}_v$ and $\phi_{n,v'} : \mathbb{S} \rightarrow \mathbb{S}_{v'}$ defined by :

$$\phi_{n,v}(z) := z \quad \text{and} \quad \phi_{n,v'}(z) := z/n \quad (\text{cf figure 8}),$$

we prove that $x_n \xrightarrow[\phi_n]{\mathcal{T}^X}$.

Lemma 3.4. *Let v and v' be two distinct internal vertices of \mathcal{T}^X and a sequence of marked spheres $(x_n)_n$ such that $x_n \xrightarrow[\phi_n]{\mathcal{T}^X}$. Then the sequence of isomorphisms $(\phi_{n,v'} \circ \phi_{n,v}^{-1})_n$ converges locally uniformly outside $i_v(v')$ to the constant $i_{v'}(v)$.*

Proof. Each vertex v and v' has three edges and every branch has at least a leaf so there exists four marked points $\chi_1, \chi_2, \chi_3, \chi_4 \in X$ such that v separates χ_1, χ_2 and v' , and the vertex v' separates χ_3, χ_4 and v .

We define for $j \in \{1, 2, 3, 4\}$,

$$\xi_j := a_v(\chi_j), \quad \xi'_j := a_{v'}(\chi_j), \quad \xi_{j,n} := \phi_{n,v} \circ x_n(\chi_j) \quad \text{and} \quad \xi'_{j,n} := \phi_{n,v'} \circ x_n(\chi_j).$$

From the hypothesis $\xi_{j,n} \rightarrow \xi_j$ and $\xi'_{j,n} \rightarrow \xi'_j$ when $n \rightarrow \infty$. Moreover, $\xi_3 = \xi_4 = i_v(v')$ and $\xi'_1 = \xi'_2 = i_{v'}(v)$. Even if we must post-compose $\phi_{n,v}$ and $\phi_{n,v'}$ by automorphisms of \mathbb{S}_v and $\mathbb{S}_{v'}$ that are converging to the identity when $n \rightarrow \infty$ and don't change the limit of $\phi_{n,v'} \circ \phi_{n,v}^{-1}$, we can suppose that for all n ,

$$\xi_{1,n} = \xi_1, \quad \xi_{2,n} = \xi_2, \quad \xi_{3,n} = \xi_3, \quad \xi'_{1,n} = \xi'_1, \quad \xi'_{3,n} = \xi'_3 \quad \text{and} \quad \xi'_{4,n} = \xi'_4.$$

Now we consider the projective charts σ on \mathbb{S}_v and σ' on $\mathbb{S}_{v'}$ defined by :

- $\sigma(\xi_1) = 0, \sigma(\xi_2) = 1$ and $\sigma(\xi_3) = \infty$;
- $\sigma'(\xi'_1) = 0, \sigma'(\xi'_4) = 1$ and $\sigma'(\xi'_3) = \infty$.

The Moebius transformation $M_n := \sigma' \circ \phi_{n,v'} \circ \phi_{n,v}^{-1} \circ \sigma^{-1}$ fixes 0 and ∞ and maps $\sigma(\xi_4)$ to 1. Thus

$$M_n(z) = \frac{z}{\lambda_n} \quad \text{with} \quad \sigma(\xi_{4,n}) \xrightarrow{n \rightarrow \infty} \infty.$$

Consequently, M_n converges locally uniformly outside infinity to the constant map equal to zero. Then, $\phi_{n,v'} \circ \phi_{n,v}^{-1} = \sigma'^{-1} \circ M_n \circ \sigma$ converges locally uniformly to the constant $(\sigma')^{-1}(0) = i_{v'}(v)$ outside $\sigma^{-1}(\infty) = i_v(v')$. \square

3.3 Convergence of marked spheres covers

To each marked rational map (f, y, z) , we can associate a cover between trees of spheres from a sphere marked by Y via the map y to a sphere marked by Z via the map z .

Recall. A portrait \mathbf{F} of degree $d \geq 2$ is a pair (F, \deg) where

- $F : Y \rightarrow Z$ is a map between finite sets Y and Z and
- $\deg : Y \rightarrow \mathbb{N} - \{0\}$ is a function verifying

$$\sum_{a \in Y} (\deg(a) - 1) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \deg(a) = d \quad \text{for all } b \in Z.$$

Definition 3.5 (Non dynamical convergence). *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres of portrait \mathbf{F} . A sequence $(f_n, y_n, z_n)_n$ of marked spheres covers converges to \mathcal{F} if their portrait is \mathbf{F} and if for all pair of internal vertices v and $w := F(v)$, there exists sequences of isomorphisms $\phi_{n,v}^Y : \mathbb{S}_n^Y \rightarrow \mathbb{S}_v$ and $\phi_{n,w}^Z : \mathbb{S}_n^Z \rightarrow \mathbb{S}_w$ such that*

- $\phi_{n,v}^Y \circ y_n : Y \rightarrow \mathbb{S}_v$ converges to $a_v^Y : Y \rightarrow \mathbb{S}_v$,
- $\phi_{n,w}^Z \circ z_n : Z \rightarrow \mathbb{S}_w$ converges to $a_w^Z : Z \rightarrow \mathbb{S}_w$ and
- $\phi_{n,w}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} : \mathbb{S}_v \rightarrow \mathbb{S}_w$ converges locally uniformly outside Y_v to $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_w$.

We use the notation $(f_n, y_n, z_n) \rightarrow \mathcal{F}$ or $(f_n, y_n, z_n) \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F}$.

Lemma 3.6. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres with portrait \mathbf{F} and of degree D . Let $v \in IV^Y$ with $\deg(v) = D$ and let $(f_n, y_n, z_n)_n$ be a sequence of marked spheres covers that satisfies $(f_n, y_n, z_n) \xrightarrow{\phi_n^Y, \phi_n^Z} \mathcal{F}$. Then the sequence $\phi_{n,F(v)}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$ converges uniformly to $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$.*

Proof. We define $w := F(v)$. We chose the projective charts $\sigma_v : \mathbb{S}_v \rightarrow \mathbb{S}$ and $\sigma_w : \mathbb{S}_w \rightarrow \mathbb{S}$ such that no any point of Y_v or of Z_w maps to infinity. We define

$$g_n := \sigma_w \circ \phi_{n,w} \circ f_n \circ \phi_{n,v}^{-1} \circ \sigma_v^{-1} \quad \text{and} \quad g := \sigma_w \circ f_v \circ \sigma_v^{-1}.$$

We supposed that the sequence $(g_n)_n$ converges locally uniformly to g out of $\sigma_v(Y_v)$. All the D poles of g_n (counting with multiplicities) converge so to the D poles of g . In particular, if U is a sufficiently small neighborhood of $\sigma_v(Y_v)$, then

- for n large enough, g_n is holomorphe without poles in U and
- $g_n - g$ converges uniformly to 0 on the boundary of U .

From the maximum modulus principle, $g_n - g$ converges uniformly to 0 in U . So g_n converge locally uniformly to g in the neighborhood of points of \mathbb{S} and given that \mathbb{S} is compact, then g_n converges uniformly to g on \mathbb{S} . \square

3.4 Dynamical convergence of marked spheres covers

Definition 3.7 (Dynamical convergence). *Let $(\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z, \mathcal{T}^X)$ be a dynamical system of trees of spheres with portrait \mathbf{F} . A sequence $(f_n, y_n, z_n)_n$ of dynamical systems between spheres marked by (\mathbf{F}, X) converges to $(\mathcal{F}, \mathcal{T}^X)$ if*

$$(f_n, y_n, z_n) \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F} \quad \text{with} \quad \phi_{n,v}^Y = \phi_{n,v}^Z$$

for all vertex $v \in IV^X$. We say that $(\mathcal{F}, \mathcal{T}^X)$ is dynamically approximable by $(f_n, y_n, z_n)_n$.

We use the notation $(f_n, y_n, z_n) \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}$ or simply $(f_n, y_n, z_n) \xrightarrow{\triangleleft} \mathcal{F}$.

We denote by $\partial \text{Rat}_{\mathbf{F}, X}$ the set of dynamical system of trees of spheres which are approximable by a sequence in $\text{Rat}_{\mathbf{F}, X}$ which are not in $\text{Rat}_{\mathbf{F}, X}$. We use the notation ϕ_n instead of ϕ_n^* when there will not be any possible confusion.

Note that requiring a dynamical convergence is not something very strong because we can prove the following:

Lemma 3.8. *If $(f_n, y_n, z_n) \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F}$ and $(\mathcal{F}, \mathcal{T}^X) \in \text{Rat}_{\mathbf{F}, X}$ then there exists $(\tilde{f}_n^Z)_n$ such that $(f_n, y_n, z_n) \xrightarrow[\phi_n^Y, \tilde{\phi}_n^Z]{\triangleleft} \tilde{\mathcal{F}}$ with $\tilde{F} = F$.*

Proof. It is sufficient to change for every $v \in IV^X$ the map $\phi_{n,v}^Z$ for $\phi_{n,v}^X$ in the collection ϕ_n^Z .

Indeed, take $w \in IV^X$. We have $a_w^X = a_w^Z|_X$ so as X_w contains at least three elements, we deduce that $\phi_{n,w}^X \circ (\phi_{n,w}^Z)^{-1}$ converges uniformly to a Moebius transformation M . Then, as for $w = F(v)$, the map $\phi_{n,w}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1}$ converges locally uniformly outside Y_v to f_v , it is the same for $\phi_{n,w}^X \circ f_n \circ (\phi_{n,v}^Y)^{-1}$ converges uniformly to $M \circ f_v$. \square

Corollary 3.9 (of lemma 3.6). *Let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres of degree d , dynamically approximable by $(f_n, y_n, z_n)_n$. Suppose that $v \in IV^X$ is a fixed vertex such that $\deg(v) = d$. Then the sequence $[f_n] \in \text{rat}_d$ converges to the conjugacy classe $[f_v] \in \text{rat}_d$.*

Lemma 3.10. *Let $(\mathcal{F}, \mathcal{T}^X) \in \text{Rat}_{\mathbf{F}, X}$ be dynamically approximable by $(f_n, y_n, z_n)_n$. If $v \in IV(F^k)$ and if $w := F^k(v)$, then $(\phi_{n,w} \circ f_n^k \circ \phi_{n,v}^{-1})_n$ converges locally uniformly to f_v^k outside a finite number of points.*

Proof. Indeed, it is sufficient to note that

$$\phi_{n,v'} \circ f_n^k \circ \phi_{n,v}^{-1} = \phi_{n,v'} \circ f_n \circ \phi_{n,F^{k-1}(v)}^{-1} \cdots \circ \phi_{n,F^2(v)} \circ f_n \circ \phi_{n,F(v)}^{-1} \circ \phi_{n,F(v)} \circ f_n \circ \phi_{n,v}^{-1}$$

so there is local uniform convergence as soon as the domain iterated does not intersect any attaching point of any edge. \square

4 Rescaling-limits

4.1 From trees of spheres to rescaling-limits

In this section we recall the rescaling limits' definition given in the introduction. These definitions are given by Jan Kiwi in [K3]. Then we explain the relation between rescaling limits and dynamical systems between trees of spheres approximable by a sequence of dynamical systems of marked spheres.

Definition 4.1 (Rescaling limits). *Let $(f_n)_n$ be a rescaling of period k and $g = \lim f_n^k$ then $[g] \in \text{rat}_{\deg g}$ is called a rescaling limit of the sequence $([f_n])_n$ in rat_d .*

Definition 4.2. *For a sequence of rational maps $(f_n)_n$ of a given degree, a rescaling is a sequence of Moebius transformations $(M_n)_n$ such that there exist $k \in \mathbb{N}$ and a rational map g of degree ≥ 2 such that*

$$M_n \circ f_n^k \circ M_n^{-1} \rightarrow g$$

uniformly in compact subsets of \mathbb{S} with finitely many points removed.

If this k is minimum then it is called the rescaling period for $(f_n)_n$ at $(M_n)_n$ and g a rescaling limit for $(f_n)_n$.

Note that naturally we are interested in sequences in rat_d so there is an equivalence relation associated to rescalings if we want to look rescaling limits in their natural quotient space $([g] \in \text{rat}_{\deg g})$ which is the one defined below.

Definition 4.3 (Independence and equivalence of rescalings). *Two rescalings $(M_n)_n$ and $(N_n)_n$ of a sequence of rational maps $(f_n)_n$ are independent if $N_n \circ M_n^{-1} \rightarrow \infty$ in Rat_1 . That is, for every compact set K in Rat_1 , the sequence $N_n \circ M_n^{-1} \notin K$ for n big enough. They are said equivalent if $N_n \circ M_n^{-1} \rightarrow M$ in Rat_1 .*

Definition (Dynamical dependence). *Given a sequence $(f_n)_n \in \text{Rat}_d$ and given $(M_n)_n$ and $(N_n)_n$ of period dividing q . We say that $(M_n)_n$ and $(N_n)_n$ are dynamically dependent if, for some subsequences $(M_{n_k})_{n_k}$ and $(N_{n_k})_{n_k}$, there exist $1 \leq m \leq q$, finite subsets S_1, S_2 of \mathbb{S} and non constant rational maps g_1, g_2 such that*

$$L_{n_k}^{-1} \circ f_{n_k}^m \circ M_{n_k} \rightarrow g_1$$

uniformly on compact subsets of $\mathbb{S} \setminus S_1$ and

$$M_{n_k}^{-1} \circ f_{n_k}^{q-m} \circ L_{n_k} \rightarrow g_2$$

uniformly on compact subsets of $\mathbb{S} \setminus S_2$.

Theorem (2). *Let \mathbf{F} be a portrait, let $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F}, X}$ and let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres. Suppose that*

$$f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}.$$

If v is a periodic internal vertex in a critical cycle with exact period k , then $f_v^k : \mathbb{S}_v \rightarrow \mathbb{S}_v$ is a rescaling limit corresponding to the rescaling $(\phi_{n,v}^Y)_n$.

In addition, for every v' in the cycle, $(\phi_{n,v'}^Y)_n$ and $(\phi_{n,v}^Y)_n$ are dynamically dependent rescalings.

Proof. If v is a periodic internal vertex in a critical cycle with exact period k then according to lemma 3.10, $(\phi_{n,v}^Y \circ f_n^k \circ (\phi_{n,v}^Y)^{-1})_n$ converges locally uniformly to $f_v^k : \mathbb{S}_v \rightarrow \mathbb{S}_v$ so $(\phi_{n,v}^Y)_n$ is a rescaling and the rescaling limit is f_v^k .

Again, according to lemma 3.10, if $0 < k' < k$, then $(\phi_{n, Fk'(v)} \circ f_n^{k'} \circ \phi_{n,v}^{-1})_n$ and $(\phi_{n,v} \circ f_n^{k-k'} \circ \phi_{n, Fk'(v)}^{-1})_n$ converge respectively locally uniformly outside finite sets to $f_v^{k'}$ and $f_{Fk'(v)}^{k-k'}$, so the rescalings $(\phi_{n,v})_n$ and $(\phi_{n, Fk'(v)})_n$ are dynamically dependent. \square

4.2 From rescaling-limits to trees of spheres

In this section, we explore the reciprocal question: if there exist rescaling limits, does there exist a dynamical system between trees of spheres such that these rescalings correspond to spheres in critical periodic cycles as described in the previous section? The following theorem gives the answer.

Theorem (3). *Given a sequence $(f_n)_n$ in Rat_d for $(d \geq 2)$ with $p \in \mathbb{N}^*$ classes M_1, \dots, M_p of rescalings. Then, passing to a subsequence, there exists a portrait \mathbf{F} , a sequence $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F}, X}$ and a dynamical system between trees of spheres \mathcal{F}_X such that*

- $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}$ and
- $\forall i \in [1, p], \exists v_i \in \mathcal{T}^Y, M_i \sim (\phi_{n,v_i}^Y)_n$.

Proof. After passing to a subsequence we can suppose that the number of critical values of the f_n and the number of their preimages and their respective multiplicities are constant.

Suppose that $\forall n \in \mathbb{N}, M_n = Id$. Denote by g the corresponding rescaling limit. The map g has at least three periodic repelling cycles. Take one point, on each of these cycles, x^1, x^2 and x^3 . As the cycles are repelling they still exists on a neighborhood of g . We can take x_n^i of fixed period $p_i \in \mathbb{N}$ such that $(x_n^i) \rightarrow x^i$. Let

- X_n be the union of the cycles of x_n^1, x_n^2 and x_n^3 ;
- Z_n be the union of X_n and the set of critical values of f_n and
- Y_n be $f_n^{-1}(Z_n)$.

After passing to a subsequence we can suppose that the cardinals of X_n, Y_n and Z_n don't depend on n . After changing the representative we can suppose that $x_0^i = x^i$. Define $x_n : X_0 \rightarrow \mathbb{S}$ by $x_n(x^i) = x_n^i$. Then, passing to a subsequence, we define y_n and z_n such that the following diagram commutes :

$$\begin{array}{ccc}
 Y_0 & \xrightarrow{y_n} & Y_n \subset \mathbb{S} \\
 \downarrow f_0 & \nearrow & \downarrow f_n \\
 & X_0 \xrightarrow{x_n} X_n & \\
 \downarrow & \nwarrow & \downarrow \\
 Z_0 & \xrightarrow{z_n} & Z_n \subset \mathbb{S}
 \end{array}$$

It follows that the (f_n, y_n, x_n) are dynamical systems between marked spheres of portrait given by the restriction of f_0 and its corresponding degree function (again after extraction). From Theorem C, there exists dynamical systems between trees of spheres $(\mathcal{F}, \mathcal{T}^X)$ which is approximable by this sequence so dynamically approximable by this sequence according to lemma 3.8.

Let v be the vertex separating x^1, x^2 and x^3 in T^Y . Using lemma 4.6, that we will prove later, we can suppose that the vertex v is not forgotten by F^k (indeed, v separate three elements of z and from this lemma we can assume that they are in T^X and then apply lemma 2.4). Define $v' := F^k(v)$.

We are going to prove that :

- $\phi_{n,v}$ converges to a Moebius transformation M ,
- $v' = v$,
- f_v^k and f_n^k are equivalents.

The first points come from the fact that $\phi_{n,v} \circ x_n(x^i) \rightarrow a_v(x^i)$. For the second point we remark that

$$\phi_{n,v} \circ \phi_{n,v'}^{-1} \circ (\phi_{n,v'} \circ f_n^k \circ \phi_{n,v}^{-1}) = \phi_{n,v} \circ f_n^k \circ \phi_{n,v}^{-1}.$$

Indeed, the right side converges to g and the term between parenthesis converges to f_v^k . We deduce that $f_v^k = g$ so the third point follows and as $\phi_{n,v}$ converges to a Moebius transformation M , we proved that $(M_n)_n = (Id)_n \sim (\phi_{n,v_i}^Y)_n$.

Suppose that $(M_n)_n$ is a rescaling of period k . As

$$(f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}) \implies (M_n \circ f_n \circ M_n^{-1} \xrightarrow[\phi_n^Y \circ M_n^{-1}, \phi_n^Z \circ M_n^{-1}]{\triangleleft} \mathcal{F}),$$

we can consider that $M_n = Id$ and use the preceding case. If we have more rescaling limits, we can adapt this proof by marking three periodic cycles for each rescaling limit. \square

First we define the following.

Definition 4.4 (Extension). *Let \tilde{X} , \tilde{Y} and \tilde{Z} be finite sets containing at least three elements with $\tilde{X} \subset \tilde{Y} \cap \tilde{Z}$. We say that $(\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z, \mathcal{T}^X)$ is an extension of $(\tilde{\mathcal{F}} : \mathcal{T}^{\tilde{Y}} \rightarrow \mathcal{T}^{\tilde{Z}}, \mathcal{T}^{\tilde{X}})$ if these are two dynamical systems between trees of spheres and if*

- $\mathcal{T}^* \triangleleft \mathcal{T}^*$,
- $\tilde{F}|_{IV^Y} = F|_{IV^Y}$ and
- $(\deg \tilde{F})|_Y = \deg F$.

We will write $(\mathcal{F}, \mathcal{T}^X) \triangleleft (\tilde{\mathcal{F}}, \mathcal{T}^{\tilde{X}})$ and more generally we use the notation $(f_n, y_n, z_n)_n \triangleleft (f_n, \tilde{y}_n, \tilde{z}_n)_n$ when for every $n \in \mathbb{N}$ we have $(f_n, y_n, z_n) \triangleleft (f_n, \tilde{y}_n, \tilde{z}_n)$ and the $(f_n, \tilde{y}_n, \tilde{z}_n)$ have the same portrait.

Before proving lemma 4.6, we first prove the following lemma.

Lemma 4.5. *If $(x_n)_n$ and $(y_n)_n$ are sequences of spheres marked respectively by X and Y such that $(x_n)_n \triangleleft (y_n)_n$ and $x_n \xrightarrow[\phi_n^X]{} \mathcal{T}^X$ then after passing to a subsequence, there exists a tree of spheres \mathcal{T}^Y such that:*

- $y_n \xrightarrow[\phi_n^Y]{} \mathcal{T}^Y$,
- $\mathcal{T}^X \triangleleft \mathcal{T}^Y$ and
- $\forall v \in T^X, \phi_{n,v}^X = \phi_{n,v}^Y$.

Proof. Using theorem B (cf introduction), after passing to a subsequence, we define a tree $\tilde{\mathcal{T}}^Y$ and a sequence $\check{\phi}_{n,v}$ for all $v \in \tilde{T}^Y$ such that $y_n \xrightarrow[\check{\phi}_n]{} \tilde{\mathcal{T}}^Y$.

Then for every vertex $v \in T^X$ separating three elements of X , we consider the vertex \check{v} in \tilde{T}^Y separating the same three elements and we want to replace the \check{v} by the v in \tilde{T}^Y to define a new tree \mathcal{T}^Y and the $\check{\phi}_{n,\check{v}}$ by the $\phi_{n,v}$ such that the lemma follows immediately. This is possible if, when two triple of elements of X separate the same vertex in T^X , then they do the same in \tilde{T}^Y .

Consider two triples $t_i = 1 = (\chi_1, \chi_2, \chi_3)$ and $t_2 = (\check{\chi}_1, \check{\chi}_2, \check{\chi}_3)$ that are separated by the same vertex v in T^X , but not in \tilde{T}^Y . After changing the

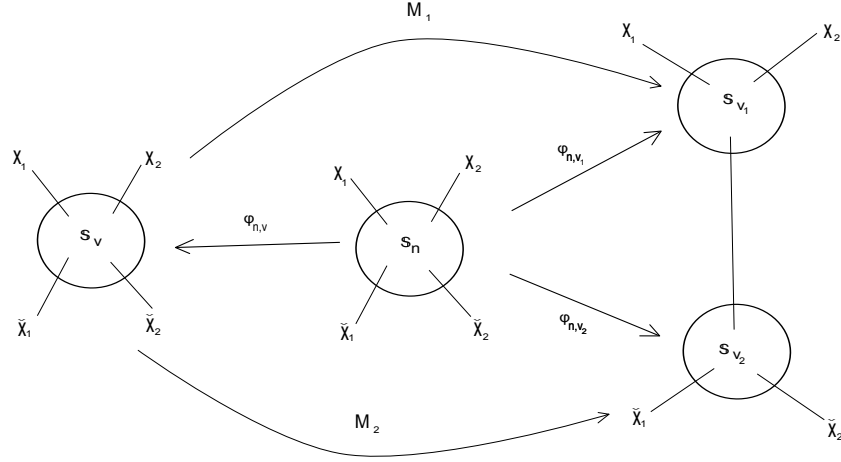


Figure 9: Notations for the proof of lemma 4.5.

labelings we can consider that v_1 separate χ_1, χ_2, χ_3 in T^Y with $i_{\tilde{v}}(\chi_1) = i_{\tilde{v}}(\chi_2)$ and that v_2 separate $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$ in T^Y with $i_v(\tilde{\chi}_1) = i_v(\tilde{\chi}_2)$ as on figure 9. Define the Moebius transformations $M_i : \mathbb{S}_v \rightarrow \mathbb{S}_{v_i}$ such that for $1 \leq j \leq 3$, M_1 maps $i_v(\chi_j)$ to $i_{v_1}(\chi_j)$ and M_1 maps $i_v(\tilde{\chi}_j)$ to $i_{v_2}(\tilde{\chi}_j)$.

Then we have $M_2 \circ M_1^{-1} = \lim(\phi_{n,v_2} \circ \phi_{n,v}^{-1}) \circ (\phi_{n,v} \circ \phi_{n,v_1}^{-1}) = \lim \phi_{n,v_2} \circ \phi_{n,v_1}^{-1}$. This would contradict lemma 3.4. \square

Lemma 4.6. Suppose that $f_n \xrightarrow{\triangleleft} \mathcal{F}$ and $z \in Z \setminus X$ then after passing to a subsequence there exist extensions $(f_n, y_n, z_n)_n \triangleleft (f_n, \tilde{y}_n, \tilde{z}_n)_n$ with $z \in \tilde{X}$ and $\forall n \in \mathbb{N}, \tilde{x}_n(z) = z_n(z)$ and $\tilde{\mathcal{F}}$ such that $\tilde{f}_n \xrightarrow{\triangleleft} \tilde{\mathcal{F}}$ and

- $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{X}}, \mathcal{T}^Y \triangleleft \mathcal{T}^{\tilde{Y}},$ and $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{Z}},$
- $\forall v \in IV^Y, F(v) \in T^X \implies \tilde{f}_v = f_v.$

Proof. After passing to a subsequence we can assume that, either there exists $y \in Y$ such that $\forall n \in \mathbb{N}, z_n(z) = y_n(y)$, or $\forall n \in \mathbb{N}, z_n(z) \notin y_n(Y)$.

In the first case we define $\tilde{X} = X \cup \{z\}$. We define $\forall n \in \mathbb{N}, \tilde{x}_n(x) = x_n(x)$ for all $x \in X$ and $\tilde{x}_n(y) = y_n(y)$; we then have $(x_n)_n \triangleleft (\tilde{x}_n)_n$. Using lemma 4.5, we define a tree $\mathcal{T}^{\tilde{X}}$. Either $\mathcal{T}^{\tilde{X}} = \mathcal{T}^X \triangleleft \mathcal{T}^Y$, or $\mathcal{T}^{\tilde{X}}$ has exactly one more vertex than \mathcal{T}^X . In the latter case this vertex v separate a (y, x_1, x_2) with $x_1, x_2 \in X$ and (y, x_1, x_2) separate a unique vertex v' in \mathcal{T}^Y . After replacing v by v' in $\mathcal{T}^{\tilde{X}}$, we have $\mathcal{T}^{\tilde{X}} \triangleleft \mathcal{T}^Y$ and the tree $\mathcal{T}^{\tilde{X}}$ still satisfies the conclusion of lemma 4.5.

We define $\tilde{Y} := Y$ and $(\tilde{y}_n)_n := (y_n)_n, \mathcal{T}^{\tilde{Y}} := \mathcal{T}^Y$ and we then have $\mathcal{T}^Y \triangleleft \mathcal{T}^{\tilde{Y}}$. We identify z and y in Z , $(\tilde{z}_n)_n$ and $(z_n)_n, \mathcal{T}^{\tilde{Z}}$ and \mathcal{T}^Z after replacing the vertex separating (y, x_1, x_2) in \mathcal{T}^Z by v' . We then identify $\tilde{\mathcal{F}}$ and \mathcal{F} according to the previous identifications and the result follows.

In the second case (ie $\forall n \in \mathbb{N}, z_n(z) \notin y_n(Y)$) we define $\tilde{X} := X \cup \{z\}$ and, using the same type of arguments as in the first case, we follow the following steps.

★ We set $\tilde{Y} = Y \cup \{z_0(z)\}$. We construct a tree $\tilde{\mathcal{T}}^Y$ with $\mathcal{T}^Y \triangleleft \mathcal{T}^{\tilde{Y}}$ by extending y_n to an injection \tilde{y}_n with $\tilde{y}_n(z) = z_n(z)$, then a tree $\tilde{\mathcal{T}}^X$ with $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{X}}$ by extending x_n to an injection \tilde{x}_n with $\tilde{x}_n(z) = z_n(z)$. After a replacement of vertex on $\mathcal{T}^{\tilde{X}}$ we can suppose that $\mathcal{T}^{\tilde{X}} \triangleleft \mathcal{T}^{\tilde{Y}}$.

★ We set $\tilde{Z} = Z \cup \{f_0(y_0(z))\}$. We construct a tree $\mathcal{T}^{\tilde{Z}}$ with $\mathcal{T}^Z \triangleleft \mathcal{T}^{\tilde{Z}}$ by extending z_n to an injection \tilde{z}_n with $\tilde{z}_n(z) = f_n(y_n(z))$. After a replacement of vertex on $\mathcal{T}^{\tilde{Z}}$ we can suppose that $\mathcal{T}^{\tilde{X}} \triangleleft \mathcal{T}^{\tilde{Z}}$. (Note that here we don't have necessarily $\mathcal{T}^Z \triangleleft \mathcal{T}^{\tilde{Z}}$.)

★ We set $\tilde{Y} = Y \cup \{f_0^{-1}(f_0(y_0(z)))\}$ construct a tree $\mathcal{T}^{\tilde{Y}}$ with $\tilde{\mathcal{T}}^Y \triangleleft \mathcal{T}^{\tilde{Y}}$ by extending \tilde{y}_n to an injection \tilde{y}_n with $\tilde{y}_n(\tilde{Y} - Y) = f_n^{-1}(f_n(z))$ such that $\mathcal{T}^{\tilde{Y}} \triangleleft \mathcal{T}^{\tilde{Z}}$.

★ Thus we have $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{X}}$, $\mathcal{T}^Y \triangleleft \mathcal{T}^{\tilde{Y}}$, and $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{Z}}$.

According to Theorem C, there exists a cover between trees of spheres $\tilde{\mathcal{F}} : \mathcal{T}^{\tilde{Y}} \rightarrow \mathcal{T}^{\tilde{Z}}$ such that $(\tilde{f}_n, \tilde{y}_n, \tilde{z}_n)_n \rightarrow \tilde{\mathcal{F}}$. Suppose that there exists a vertex $v \in IV^Y$ such that $F(v) = v' \in T^X$ and $\tilde{F}(v) = v'' \in T^{\tilde{Z}}$ with $v' \neq v''$. Then $v' \in T^{\tilde{Z}}$ because $\mathcal{T}^X \triangleleft \mathcal{T}^{\tilde{Z}}$. Thus $\phi_{n,v'} \circ \phi_{n,v''}^{-1}$ converges uniformly outside a finite number of points to a constant. However,

$$f_v = \lim \phi_{n,v'} \circ f_n \circ \phi_{n,v}^{-1} = \lim (\phi_{n,v'} \circ \phi_{n,v''}^{-1}) \circ \phi_{n,v''} \circ f_n \circ \phi_{n,v}^{-1}$$

but $\phi_{n,v''} \circ f_n \circ \phi_{n,v}^{-1}$ converges uniformly to \tilde{f}_v outside a finite set so this is impossible and $v' = v''$. \square

4.3 Theorem A and further considerations

Proof. [Theorem A] Take a sequence $(f_n)_n$ in Rat_d for $d \geq 2$ and suppose that it has strictly more then $p > 2d - 2$ dynamically independent rescalings for which the associated rescaling limits are non post-critically finite. Then according to Theorem 4.2, passing to a subsequence, there exists a portrait \mathbf{F} , a sequence $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F},X}$ and a dynamical system between trees of spheres $(\mathcal{F}, \mathcal{T}^X)$ such that

$$f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F},$$

thus according to Theorem 4.1 these classes of dynamically independent rescalings are associated to critic periodic cycles of spheres with a non post-critically finite associated cover. As \mathbf{F} has degree d because the f_n lie in Rat_d this contradicts corollary 2.3. \square

We can see from the proof of Theorem 4.2 that it is sufficient to mark some cycles to find the rescaling-limits but there is still an important question.

Question 4.7. *How to know which cycles we have to mark in order to find the rescaling-limits?*

In general this is not simple. For example, in [EP], the authors are proving that we can have a non-trivial rescaling of any period in the case of degree 2. There is an other question that the reader should keep in mind. We defined dynamical systems between trees of spheres in a very general setting but the one that lye to an interpretation in terms of rescaling limits are the one witch are dynamically approximable by some sequence of dynamically marked rational maps. So we naturally ask the following:

Question 4.8. *Is every dynamical system between trees of spheres dynamically approximable by some sequence of dynamically marked rational maps?*

The answer to this question is no and a counterexample is given in [A]. This answer require more technical results that will be made explicit in [A1] where we will give some necessary conditions for a dynamical systems between trees of spheres to be approximable by some sequence of dynamically marked rational maps.

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